

Smoothness of Generalized Solutions for Nonlocal Elliptic Problems on the Plane

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Abstract

We study smoothness of generalized solutions of nonlocal elliptic problems in plane bounded domains with piecewise smooth boundary. The case where the support of nonlocal terms can intersect the boundary is considered. We announce conditions that are necessary and sufficient for any generalized solution to possess an appropriate smoothness (in terms of Sobolev spaces). The proofs are given in the forthcoming paper.

1. The most difficult situation in the theory of elliptic problems with nonlocal boundary-value conditions is that where the support of nonlocal terms can intersect a boundary of a domain [1]–[8]. In that case, solutions can have power-law singularities near some points on the boundary. In the present paper, we find out conditions that are necessary and sufficient for any generalized solution $u \in W_2^1(G)$ of a nonlocal problem in a plane bounded domain G to belong to $W_2^2(G)$. We study the case in which different nonlocal conditions are set on different parts of the boundary, coefficients of nonlocal terms supported near the points of conjugation of boundary conditions are variable, and nonlocal operators corresponding to nonlocal terms supported outside the conjugation points are abstract. Both homogeneous and nonhomogeneous nonlocal conditions are investigated. We consider a nonlocal perturbation of the Dirichlet problem for an elliptic equation of order two. However, the obtained results can be generalized to elliptic equations of order $2m$ with general nonlocal conditions.

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G . Introduce a set $\mathcal{K} \subset \partial G$ consisting of finitely many points. Let $\partial G \setminus \mathcal{K} = \bigcup_{i=1}^N \Gamma_i$, where Γ_i are open (in the topology of ∂G) C^∞ -curves. We assume that the domain G is a plane angle in some neighborhood of each point $g \in \mathcal{K}$. Denote by \mathbf{P} a differential operator of order two, with smooth complex-valued coefficients, properly elliptic in \overline{G} .

For any closed set \mathcal{M} , write $\mathcal{O}_\varepsilon(\mathcal{M}) = \{y \in \mathbb{R}^2 : \text{dist}(y, \mathcal{M}) < \varepsilon\}$, where $\varepsilon > 0$.

We now define operators corresponding to nonlocal conditions near the set \mathcal{K} . Let Ω_{is} ($i = 1, \dots, N$; $s = 1, \dots, S_i$) denote C^∞ -diffeomorphisms taking a neighborhood \mathcal{O}_i of the curve $\overline{\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})}$ onto the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$ and $\Omega_{is}(g) \in \mathcal{K}$ for $g \in \overline{\Gamma_i \cap \mathcal{K}}$. Thus, the transformations Ω_{is} take the curves $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ strictly inside the domain G and their end points $\overline{\Gamma_i \cap \mathcal{K}}$ to the end points.

Let us specify the structure of the transformations Ω_{is} near the set \mathcal{K} . Denote by Ω_{is}^{+1} the transformation $\Omega_{is} : \mathcal{O}_i \rightarrow \Omega_{is}(\mathcal{O}_i)$ and by $\Omega_{is}^{-1} : \Omega_{is}(\mathcal{O}_i) \rightarrow \mathcal{O}_i$ the transformation inverse to Ω_{is} . The set of the points $\Omega_{i_q s_q}^{\pm 1}(\dots \Omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}$ ($1 \leq s_j \leq S_{i_j}$, $j = 1, \dots, q$) is called an *orbit* of the point $g \in \mathcal{K}$. In other words, the orbit of $g \in \mathcal{K}$ is formed by the points that can be obtained by consecutively applying the transformations $\Omega_{is}^{\pm 1}$ to g . We assume for simplicity that the set $\mathcal{K} = \{g_1, \dots, g_N\}$ consists of one orbit only.

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Let ε be small so that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$ of the points $g_j \in \mathcal{K}$ satisfying the following conditions: (I) $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_{\varepsilon}(g_j)$, (II) the boundary ∂G is an angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$, (III) $\mathcal{O}_{\varepsilon_1}(g_j) \cap \mathcal{O}_{\varepsilon_1}(g_k) = \emptyset$ for any $g_j, g_k \in \mathcal{K}$, $k \neq j$, (IV) if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_{\varepsilon}(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}(\mathcal{O}_{\varepsilon}(g_j)) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$, fix a transformation $y \mapsto y'(g_j)$ which is a composition of the shift by the vector $-\overrightarrow{Og_j}$ and the rotation through some angle so that the set $\mathcal{O}_{\varepsilon_1}(g_j)$ is taken onto a neighborhood $\mathcal{O}_{\varepsilon_1}(0)$ of the origin, whereas $G \cap \mathcal{O}_{\varepsilon_1}(g_j)$ and $\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)$ are taken to the intersection of a plane angle $K_j = \{y \in \mathbb{R}^2 : r > 0, |\omega| < \omega_j\}$ with $\mathcal{O}_{\varepsilon_1}(0)$ and to the intersection of the side $\gamma_{j\sigma} = \{y \in \mathbb{R}^2 : \omega = (-1)^\sigma \omega_j\}$ ($\sigma = 1$ or $\sigma = 2$) of the angle K_j with $\mathcal{O}_{\varepsilon_1}(0)$, respectively. Here (ω, r) are the polar coordinates, $0 < \omega_j < \pi$.

Condition 1. *The above change of variables $y \mapsto y'(g_j)$ for $y \in \mathcal{O}_{\varepsilon}(g_j)$, $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$, reduces the transformation $\Omega_{is}(y)$ to the composition of a rotation and a homothety in the new variables y' .*

Introduce the nonlocal operators \mathbf{B}_i^1 by the formula $\mathbf{B}_i^1 u = \sum_{s=1}^{S_i} b_{is}(y) u(\Omega_{is}(y))$, $y \in \Gamma_i \cap \mathcal{O}_{\varepsilon}(\mathcal{K})$, $\mathbf{B}_i^1 u = 0$, $y \in \Gamma_i \setminus (\Gamma_i \cap \mathcal{O}_{\varepsilon}(\mathcal{K}))$, where $b_{is} \in C^\infty(\mathbb{R}^2)$ and $\text{supp } b_{is} \subset \mathcal{O}_{\varepsilon}(\mathcal{K})$. Since $\mathbf{B}_i^1 u = 0$ whenever $\text{supp } u \subset \overline{G \setminus \mathcal{O}_{\varepsilon_1}(\mathcal{K})}$, we say that the operators \mathbf{B}_i^1 correspond to nonlocal terms supported near the set \mathcal{K} .

Consider the operators \mathbf{B}_i^2 satisfying the following condition (cf. (2.5), (2.6) in [2] and (3.4), (3.5) in [6]).

Condition 2. *There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\rho > 0$ such that the inequalities*

$$\|\mathbf{B}_i^2 u\|_{W_2^{3/2}(\Gamma_i)} \leq c_1 \|u\|_{W_2^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K}))}}, \quad \|\mathbf{B}_i^2 u\|_{W_2^{3/2}(\Gamma_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K}))}} \leq c_2 \|u\|_{W_2^2(G_\rho)} \quad (1)$$

hold for any $u \in W_2^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K}))} \cap W_2^2(G_\rho)$, where $G_\rho = \{y \in G : \text{dist}(y, \partial G) > \rho\}$, $i = 1, \dots, N$, $c_1, c_2 > 0$.

In particular, the first inequality in (1) means that $\mathbf{B}_i^2 u = 0$ whenever $\text{supp } u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. Therefore, we say that the operators \mathbf{B}_i^2 correspond to nonlocal terms supported outside the set \mathcal{K} . Examples of the operators \mathbf{B}_i^2 can be found in [2, 5].

We assume that Conditions 1 and 2 are fulfilled throughout the paper.

Consider the following nonlocal elliptic boundary-value problem:

$$\mathbf{P}u = f_0(y) \quad (y \in G), \quad (2)$$

$$u|_{\Gamma_i} + \mathbf{B}_i^1 u + \mathbf{B}_i^2 u = f_i(y) \quad (y \in \Gamma_i; i = 1, \dots, N). \quad (3)$$

Denote $\mathcal{W}_2^{k-1/2}(\partial G) = \prod_{i=1}^N W_2^{k-1/2}(\Gamma_i)$ for $k \in \mathbb{N}$. For any set $X \in \mathbb{R}^2$ having a nonempty interior, we denote by $C_0^\infty(X)$ the set of functions infinitely differentiable in \overline{X} and supported in X .

Definition 1. A function $u \in W_2^1(G)$ is called a *generalized solution* of problem (2), (3) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}_2^{1/2}(\partial G)$ if u satisfies nonlocal conditions (3) (where the equalities are understood as those in $W_2^{1/2}(\Gamma_i)$) and Eq. (2) in the sense of distributions.

We now write a model nonlocal problem corresponding to the points of the set (orbit) \mathcal{K} . Denote by $u_j(y)$ the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_1}(g_j)$. If $g_j \in \overline{\Gamma_i}$, $y \in \mathcal{O}_{\varepsilon}(g_j)$, and $\Omega_{is}(y) \in \mathcal{O}_{\varepsilon_1}(g_k)$, then denote by $u_k(\Omega_{is}(y))$ the function $u(\Omega_{is}(y))$. In that case, nonlocal problem (2), (3) acquires the following form in the ε -neighborhood of the set (orbit) \mathcal{K} :

$$\mathbf{P}u_j = f_0(y) \quad (y \in \mathcal{O}_{\varepsilon}(g_j) \cap G),$$

$$u_j(y)|_{\mathcal{O}_{\varepsilon}(g_j) \cap \Gamma_i} + \sum_{s=1}^{S_i} b_{is}(y) u_k(\Omega_{is}(y))|_{\mathcal{O}_{\varepsilon}(g_j) \cap \Gamma_i} = \psi_i(y) \\ (y \in \mathcal{O}_{\varepsilon}(g_j) \cap \Gamma_i; i \in \{1 \leq i \leq N : g_j \in \overline{\Gamma_i}\}; j = 1, \dots, N),$$

where $\psi_i = f_i - \mathbf{B}_i^2 u$. Let $y \mapsto y'(g_j)$ be the above change of variables. Set $K_j^{\varepsilon} = K_j \cap \mathcal{O}_{\varepsilon}(0)$, $\gamma_{j\sigma}^{\varepsilon} = \gamma_{j\sigma} \cap \mathcal{O}_{\varepsilon}(0)$ and introduce the functions

$$U_j(y') = u_j(y(y')), F_j(y') = f_0(y(y')), y' \in K_j^{\varepsilon}, \quad \Psi_{j\sigma}(y') = \psi_i(y(y')), y' \in \gamma_{j\sigma}^{\varepsilon}, \quad (4)$$

where $\sigma = 1$ ($\sigma = 2$) if the transformation $y \mapsto y'(g_j)$ takes Γ_i to the side γ_{j1} (γ_{j2}) of the angle K_j . In what follows, we write y instead of y' . Using Condition 1, we can write problem (2), (3) as follows:

$$\mathbf{P}_j U_j = F_j(y) \quad (y \in K_j^{\varepsilon}), \quad (5)$$

$$\mathbf{B}_{j\sigma} U \equiv \sum_{k,s} b_{j\sigma ks}(y) U_k(\mathcal{G}_{j\sigma ks} y) = \Psi_{j\sigma}(y) \quad (y \in \gamma_{j\sigma}^{\varepsilon}). \quad (6)$$

Here (and below unless otherwise stated) $j, k = 1, \dots, N$; $\sigma = 1, 2$; $s = 0, \dots, S_{j\sigma k}$; \mathbf{P}_j is an elliptic differential operator of order two with smooth coefficients; $U = (U_1, \dots, U_N)$; $b_{j\sigma ks}(y)$ are smooth functions, $b_{j\sigma j0}(y) \equiv 1$; $\mathcal{G}_{j\sigma ks}$ is the operator of rotation through an angle $\omega_{j\sigma ks}$ and of homothety with a coefficient $\chi_{j\sigma ks} > 0$ in the y -plane. Moreover, $|(-1)^{\sigma} \omega_j + \omega_{j\sigma ks}| < \omega_k$ for $(k, s) \neq (j, 0)$ and $\omega_{j\sigma j0} = 0$, $\chi_{j\sigma j0} = 1$ (i.e., $\mathcal{G}_{j\sigma j0} y \equiv y$).

Write the principal parts of the operators \mathbf{P}_j at the point $y = 0$ in polar coordinates, $r^{-2} \tilde{\mathcal{P}}_j(\omega, \partial/\partial\omega, r\partial/\partial r)$. Consider the analytic operator-valued function $\tilde{\mathcal{L}}(\lambda) : \prod_j W_2^2(-\omega_j, \omega_j) \rightarrow \prod_j (L_2(-\omega_j, \omega_j) \times \mathbb{C}^2)$ given by $\tilde{\mathcal{L}}(\lambda)\varphi = \{\tilde{\mathcal{P}}_j(\omega, \partial/\partial\omega, i\lambda)\varphi_j, \sum_{k,s} (\chi_{j\sigma ks})^{i\lambda} b_{j\sigma ks}(0)\varphi_k((-1)^{\sigma}\omega_j + \omega_{j\sigma ks})\}$. Main definitions

and facts concerning analytic operator-valued functions can be found in [9]. It is fundamental that the spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete and, for any numbers $c_1 < c_2$, the band $c_1 < \text{Im } \lambda < c_2$ contains at most finitely many eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ (see [4]). Spectral properties of the operator $\tilde{\mathcal{L}}(\lambda)$ play a crucial role in the study of smoothness of generalized solutions.

2. Let $\lambda = \lambda_0$ be an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$.

Definition 2. We say that λ_0 is a *proper eigenvalue* if none of the eigenvectors $\varphi(\omega) = (\varphi_1(\omega), \dots, \varphi_N(\omega))$ corresponding to λ_0 has an associated vector, whereas the functions $r^{i\lambda_0} \varphi_j(\omega)$, $j = 1, \dots, N$, are polynomials in y_1, y_2 . An eigenvalue which is not proper is said to be *improper*.

The notion of proper eigenvalue was originally proposed by Kondrat'ev [10] for "local" boundary-value problems in nonsmooth domains.

Theorem 1. 1. Let the band $-1 \leq \operatorname{Im} \lambda < 0$ contain no eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$, and let $u \in W_2^1(G)$ be a generalized solution of problem (2), (3) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}_2^{3/2}(\partial G)$. Then $u \in W_2^2(G)$.

2. Let the band $-1 \leq \operatorname{Im} \lambda < 0$ contain an improper eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$. Then there exists a generalized solution $u \in W_2^1(G)$ of problem (2), (3) with certain right-hand side $\{f_0, 0\}$, $f_0 \in L_2(G)$, such that $u \notin W_2^2(G)$.

It remains to study the case in which the following condition holds.

Condition 3. The band $-1 \leq \operatorname{Im} \lambda < 0$ contains a unique eigenvalue $\lambda = -i$ of the operator $\tilde{\mathcal{L}}(\lambda)$, and this eigenvalue is a proper one.

We first consider problem (2), (3) with nonhomogeneous nonlocal conditions.

Denote by $\tau_{j\sigma}$ the unit vector co-directed with the ray $\gamma_{j\sigma}$. Consider the operators $\frac{\partial}{\partial \tau_{j\sigma}} \left(\sum_{k,s} b_{j\sigma ks}(0) U_k(\mathcal{G}_{j\sigma ks} y) \right)$. Using the chain rule, we write them as follows:

$$\sum_{k,s} (\hat{B}_{j\sigma ks}(D_y) U_k)(\mathcal{G}_{j\sigma ks} y), \quad (7)$$

where $\hat{B}_{j\sigma ks}(D_y)$ are first-order differential operators with constant coefficients. In particular, we have $\hat{B}_{j\sigma j0}(D_y) = \partial / \partial \tau_{j\sigma}$ because $\mathcal{G}_{j\sigma j0} y \equiv y$. Formally replacing the nonlocal operators by the corresponding local ones in (7), we introduce the operators

$$\hat{\mathcal{B}}_{j\sigma}(D_y) U \equiv \sum_{k,s} \hat{B}_{j\sigma ks}(D_y) U_k(y). \quad (8)$$

If Condition 3 holds, then the system of operators (8) is linearly dependent [11]. Let

$$\{\hat{\mathcal{B}}_{j'\sigma'}(D_y)\} \quad (9)$$

be a maximal linearly independent subsystem of system (8). In that case, any operator $\hat{\mathcal{B}}_{j\sigma}(D_y)$ which does not enter system (9) can be represented as follows:

$$\hat{\mathcal{B}}_{j\sigma}(D_y) = \sum_{j',\sigma'} \beta_{j\sigma}^{j'\sigma'} \hat{\mathcal{B}}_{j'\sigma'}(D_y), \quad (10)$$

where $\beta_{j\sigma}^{j'\sigma'}$ are some constants. Let $Z_{j\sigma} \in W_2^{3/2}(\gamma_{j\sigma}^\varepsilon)$ be arbitrary functions. Set $Z_{j\sigma}^0(r) = Z_{j\sigma}(y)|_{y=(r \cos \omega_j, r(-1)^\sigma \sin \omega_j)}$. It is clear that $Z_{j\sigma}^0 \in W_2^{3/2}(0, \varepsilon)$.

Definition 3. Let $\beta_{j'\sigma'}$ be the constants occurring in (10). If the relations

$$\int_0^\varepsilon r^{-1} \left| \frac{d}{dr} \left(Z_{j\sigma}^0 - \sum_{j',\sigma'} \beta_{j\sigma}^{j'\sigma'} Z_{j'\sigma'}^0 \right) \right|^2 dr < \infty \quad (11)$$

hold for all indices j, σ corresponding to the operators of system (8) which do not enter system (9), then we say that the functions $Z_{j\sigma}$ satisfy the consistency condition (11).

Let us formulate conditions which ensure that generalized solutions are smooth. We first show that right-hand sides f_i in nonlocal conditions (3) cannot be arbitrary functions from the space $W_2^{3/2}(\Gamma_i)$.

Consider the change of variables $y \mapsto y'(g_j)$ described in Sec. 1. Introduce the functions

$$F_{j\sigma}(y') = f_i(y(y')), \quad y' \in \gamma_{j\sigma}^\varepsilon$$

(cf. functions (4)). Denote by $\mathcal{S}_2^{3/2}(\partial G)$ the set consisting of functions $\{f_i\} \in \mathcal{W}_2^{3/2}(\partial G)$ such that the functions $F_{j\sigma}$ satisfy the consistency condition (11). The set $\mathcal{S}_2^{3/2}(\partial G)$ is not closed in the topology of $\mathcal{W}_2^{3/2}(\partial G)$ (see. [11, Lemma 3.2]).

Lemma 1. *Let Condition 3 hold. Then there exist a function $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}_2^{3/2}(\partial G)$, $\{f_i\} \notin \mathcal{S}_2^{3/2}(\partial G)$, and a function $u \in W_2^1(G)$ such that u is a generalized solution of problem (2), (3) with right-hand side $\{f_0, f_i\}$ and $u \notin W_2^2(G)$.*

It follows from Lemma 1 that, if one wants *any* generalized solution of problem (2), (3) be smooth, then one must take right-hand sides $\{f_0, f_i\}$ from the space $L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$.

Let $v \in W_2^2(G \setminus \overline{\mathcal{O}_{\mathcal{N}_1}(\mathcal{K})})$ be an arbitrary function. Consider the change of variables $y \mapsto y'(g_j)$ from Sec. 1 again and introduce the functions

$$B_{j\sigma}^v(y') = (\mathbf{B}_i^2 v)(y(y')), \quad y' \in \gamma_{j\sigma}^\varepsilon.$$

Condition 4. *For any function $v \in W_2^2(G \setminus \overline{\mathcal{O}_{\mathcal{N}_1}(\mathcal{K})})$ and for any constant vector $C = (C_1, \dots, C_N)$, the functions $B_{j\sigma}^v$ and $\mathbf{B}_{j\sigma} C$, respectively, satisfy the consistency condition (11).*

Theorem 2. *Let Condition 3 be fulfilled. Then:*

1. *If Condition 4 holds and $u \in W_2^1(G)$ is a generalized solution of problem (2), (3) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$, then $u \in W_2^2(G)$.*
2. *If Condition 4 fails, then there exists a generalized solution $u \in W_2^1(G)$ of problem (2), (3) with certain right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$ such that $u \notin W_2^2(G)$.*

We now consider problem (2), (3) with homogeneous nonlocal conditions.

Definition 4. We say that a function $v \in W_2^2(G \setminus \overline{\mathcal{O}_{\mathcal{N}_1}(\mathcal{K})})$ is *admissible* if there exists a constant vector $C = (C_1, \dots, C_N)$ such that

$$B_{j\sigma}^v(0) + (\mathbf{B}_{j\sigma} C)(0) = 0, \quad j = 1, \dots, N, \quad \sigma = 1, 2. \quad (12)$$

Any vector C satisfying relations (12) is called an *admissible vector corresponding to the function v* .

The set of admissible functions is linear. Clearly, the function $v = 0$ is an admissible function, whereas the vector $C = 0$ is an admissible vector corresponding to $v = 0$. Moreover, one can verify that any generalized solution of problem (2), (3) with homogeneous nonlocal conditions is an admissible function.

Consider the following condition (which is weaker than Condition 4).

Condition 4'. *For any admissible function v and for any admissible vector C corresponding to v , the functions $B_{j\sigma}^v + \mathbf{B}_{j\sigma} C$ satisfy the consistency condition (11).*

Theorem 2'. *Let Condition 3 be fulfilled. Then:*

1. *If Condition 4' holds and $u \in W_2^1(G)$ is a generalized solution of problem (2), (3) with right-hand side $\{f_0, 0\}$, $f_0 \in L_2(G)$, then $u \in W_2^2(G)$.*
2. *If Condition 4' fails, then there exists a generalized solution $u \in W_2^1(G)$ of problem (2), (3) with certain right-hand side $\{f_0, 0\}$, $f_0 \in L_2(G)$, such that $u \notin W_2^2(G)$.*

The proofs of Theorems 1, 2, and 2' are based on results concerning the solvability of model nonlocal problems in plane angles in Sobolev spaces [11] and on asymptotic behavior of solutions of these problems in weighted spaces [2, 12].

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